

FULLY IMPLICIT FINITE VOLUME SCHEME FOR TRANSIENT CONDUCTIVE-RADIATIVE HEAT TRANSFER: DISCRETE EXISTENCE, UNIQUENESS, MAXIMUM PRINCIPLE

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ABSTRACT. This article studies a fully implicit finite volume scheme for transient nonlinear heat transport equations coupled by nonlocal interface conditions modeling diffuse-gray radiation between the surfaces of (both open and closed) cavities. The model is considered in three space dimensions; modifications for the axisymmetric case are indicated. Extending the results of [9], where a similar, but not fully implicit, finite volume scheme was considered, a discrete maximum principle is established, yielding discrete L^∞ - L^∞ a priori bounds as well as a unique discrete solution to the finite volume scheme.

1. Introduction. Modeling and numerical simulation of conductive-radiative heat transfer has become a standard tool to aid and improve numerous industrial processes such as crystal growth by the Czochralski method [4, 8] and by the physical vapor transport method [10] to mention just two examples.

Heat transfer models including diffuse-gray radiative interactions between cavity surfaces consist of nonlinear elliptic (stationary) or parabolic (transient) PDE (heat equations), where a nonlocal coupling occurs due to the integral operator of the radiosity equation. Recent papers regarding the mathematical theory of existence, uniqueness, and regularity of weak solutions include [1, 2, 3, 6]. Discretization methods in the context of radiative heat transfer between surfaces of nonconvex cavities have been studied in [9, 13]. In [13], a finite element approximation is considered for a stationary conductive-radiative heat transfer problem. In [9], transient heat transport is treated and, in contrast to [13], heat conduction is also considered inside closed cavities, with a jumping diffusion coefficient at the interface. Moreover, the emissivity is allowed to depend on the temperature (i.e. on the solution). The setting of [9] is also used in the following.

The finite volume scheme presented in the current paper is similar to that of [9], however, it differs from the scheme in [9] in being *fully implicit*: The scheme in [9] is an implicit scheme with the exception that the temperature-dependent emissivities are approximated explicitly, i.e. they are evaluated at the temperature of the previous time step. For the scheme presented in the current paper, the emissivities are

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also approximated implicitly, i.e. they are evaluated at the temperature of the current time step. The advantage of the scheme in [9], i.e. of the explicit discretization of the emissivities, is a simpler implementation: The discrete nonlinear system is solved via Newton's method, and approximating the emissivities explicitly considerably simplifies the computation of the derivative of the discrete nonlinear operator. However, it is well-known that explicitly discretized terms in transient heat transfer problems can impair the convergence properties of the scheme by convergence requiring additional smallness conditions on the time step size k depending on the size h of the elements of the space discretization. Proving the convergence of the scheme of [9] as well as of the scheme of the present paper to the weak solution of the corresponding continuous problem does not seem easy and is still work in progress. However, preliminary results indicate the convergence proof for the [9] scheme requires an additional smallness condition of the form $k \sim h^2$ on the time step size arising from the explicitly discretized terms; and that this smallness condition can be avoided for the fully implicit scheme of the present paper. The discrete existence results of [9] as well as of the present paper only require a smallness condition of the form $k \sim h$.

Therefore, the goal of the present paper is to extend the results of [9], i.e. the discrete maximum principle as well as existence and uniqueness of a solution to the discrete scheme, to the fully implicit scheme. Both finite volume schemes lead to nonlinear and nonlocal systems of equations, the solvability of which is not at all obvious. Using an additional regularity assumption for the emissivity function, namely local Lipschitzness, the main results can be proved by the same method as in [9]: The proof of the discrete maximum principle as well as existence and uniqueness are based on the root problem with maximum principle [9, Th. 4.1].

The paper is organized as follows: In Sec. 2, the governing equations of transient conductive heat transfer are recalled, completed by nonlocal interface and boundary conditions arising from the modeling of diffuse-gray radiation. Section 2 also provides the precise mathematical setting. The discrete scheme is stated in Sec. 3, where the nonlocal radiation operators are discretized in Sec. 3.2, also providing some important properties of the resulting discrete nonlocal operators. The proof of the discrete maximum principle as well as existence and uniqueness of a discrete solution to the finite volume scheme are the subject of Sec. 4. The main results are found in Th. 4.2 and its two corollaries.

2. Transient Heat Transport Including Conduction and Diffuse-Gray Radiation.

2.1. Transient Heat Equations. Transient conductive-radiative heat transport is considered on a time-space cylinder $[0, T] \times \overline{\Omega}$, where the space domain $\Omega \subseteq \mathbb{R}^3$ is assumed to consist of two parts Ω_s and Ω_g , Ω_s representing an opaque solid and Ω_g representing a transparent gas. More precisely, we assume:

- (A-1) $T \in \mathbb{R}^+$, $\overline{\Omega} = \overline{\Omega_s} \cup \overline{\Omega_g}$, $\Omega_s \cap \Omega_g = \emptyset$, and each of the sets Ω , Ω_s , Ω_g , is a nonvoid, polyhedral, bounded, and open subset of \mathbb{R}^3 .
- (A-2) Ω_g is enclosed by Ω_s , i.e. $\partial\Omega_s = \partial\Omega \dot{\cup} \partial\Omega_g$, where $\dot{\cup}$ denotes a disjoint union. Thus, $\Sigma := \partial\Omega_g = \overline{\Omega_s} \cap \overline{\Omega_g}$, and $\partial\Omega = \partial\Omega_s \setminus \Sigma$ (see Fig. 1).

Heat conduction is considered throughout $\overline{\Omega}$. Nonlocal radiative heat transport is considered between points on the surface Σ of Ω_g as well as between points on the surfaces of open cavities (such as O_1 and O_2 in Fig. 1). However, to avoid

introducing additional boundary conditions, open cavities are not part of $\overline{\Omega}$, i.e. heat conduction is *not* considered in open cavities (see Sec. 2.3 below for details).

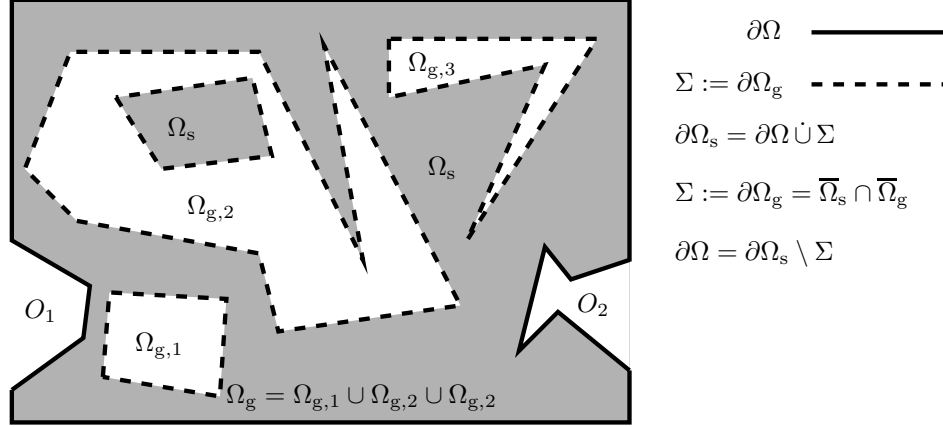


FIGURE 1. Possible shape of a 2-dimensional section through the 3-dimensional domain $\overline{\Omega} = \overline{\Omega_s} \cup \overline{\Omega_g}$ with open cavities O_1 and O_2 . Note that, according to (A-2), Ω_g is engulfed by Ω_s , which can not be seen in the 2-dimensional section.

Transient heat conduction is described by

$$\frac{\partial \varepsilon_m(\theta)}{\partial t} - \operatorname{div}(\kappa_m \nabla \theta) = f_m(t, x) \quad \text{in }]0, T[\times \Omega_m \quad (m \in \{s, g\}), \quad (1)$$

where $\theta(t, x) \in \mathbb{R}_0^+$ represents absolute temperature, depending on the time coordinate t and on the space coordinate x ; the continuous, strictly increasing, nonnegative functions $\varepsilon_m \in C(\mathbb{R}_0^+, \mathbb{R}_0^+)$ represent the internal energy in the solid and in the gas, respectively, $\kappa_m \in \mathbb{R}_0^+$ represent the thermal conductivity in solid and gas, respectively, assumed constant for simplicity, and f_m represent heat sources due to some heating mechanism. In practice, for many heating mechanisms such as induction or resistance heating, one has $f_g = 0$.

Throughout this paper, (A-3) – (A-5) are assumed, where:

(A-3) For $m \in \{s, g\}$, $\varepsilon_m : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ is continuous and at least of linear growth, i.e. there is $C_\varepsilon \in \mathbb{R}^+$ such that

$$\varepsilon_m(\theta_2) \geq (\theta_2 - \theta_1) C_\varepsilon + \varepsilon_m(\theta_1) \quad (\theta_2 \geq \theta_1 \geq 0).$$

(A-4) For $m \in \{s, g\}$: $\kappa_m \in \mathbb{R}_0^+$.

(A-5) For $m \in \{s, g\}$: $f_m \in L^\infty(]0, T[\times \Omega_m)$, $f_m \geq 0$ a.e.

2.2. Nonlocal Interface Conditions. Continuity of the heat flux on the interface Σ between solid and gas, where one needs to account for radiosity R and for irradiation J , yields the following interface condition for (1):

$$(\kappa_g \nabla \theta) \upharpoonright_{\overline{\Omega_g}} \cdot \mathbf{n}_g + R(\theta) - J(\theta) = (\kappa_s \nabla \theta) \upharpoonright_{\overline{\Omega_s}} \cdot \mathbf{n}_g \quad \text{on }]0, T[\times \Sigma. \quad (2)$$

Here, \mathbf{n}_g denotes the unit normal vector pointing from gas to solid and \upharpoonright denotes restriction (or trace).

As the solid is assumed opaque, $R(\theta)$ and $J(\theta)$ are computed according to the net radiation model for diffuse-gray surfaces, i.e. reflection and emittance are taken

to be independent of the angle of incidence and independent of the wavelength. At each point of the surface Σ , the radiosity is the sum of the emitted radiation $E(\theta)$ and of the reflected radiation $J_r(\theta)$:

$$R = E + J_r \text{ on } \Sigma. \quad (3)$$

According to the Stefan-Boltzmann law,

$$E(\theta) = \sigma \epsilon(\theta) \theta^4 \text{ on } \Sigma, \quad (4)$$

where σ represents the Boltzmann radiation constant, and ϵ represents the potentially temperature-dependent emissivity of the solid surface. It is assumed that:

(A-6) $\sigma \in \mathbb{R}^+$, $\epsilon : \mathbb{R}_0^+ \rightarrow]0, 1]$ is locally Lipschitz continuous, i.e., for each $r \in \mathbb{R}_0^+$:

$$L_{\epsilon,r} := \sup \left\{ \frac{|\epsilon(\theta_1) - \epsilon(\theta_2)|}{|\theta_1 - \theta_2|} : (\theta_1, \theta_2) \in [0, r]^2, \theta_1 \neq \theta_2 \right\} \in \mathbb{R}_0^+.$$

Remark 1. Condition (A-6) is stronger than [9, (A-6)], where ϵ was only assumed to be continuous, but not necessarily locally Lipschitz. For the proof of [9, Lem. 3.2(c)], i.e. for the proof of the local Lipschitz continuity of the maps $V_{\Gamma,\alpha}(\tilde{\mathbf{u}}, \cdot)$ and $V_{\Sigma,\alpha}(\tilde{\mathbf{u}}, \cdot)$, respectively, due to the explicit discretization of ϵ , the regularity of ϵ was not an issue. However, here, where ϵ is discretized implicitly, local Lipschitz continuity of ϵ is necessary to proof the respective local Lipschitz continuity of $V_{\Gamma,\alpha}$ and $V_{\Sigma,\alpha}$ in Lem. 3.2(c) below (in general, if ϵ is only continuous, then local Lipschitz continuity of $V_{\Gamma,\alpha}$ and $V_{\Sigma,\alpha}$ can not be expected).

Remark 2. For each $r \in \mathbb{R}^+$, the continuous map ϵ attains its minimum, denoted by $\epsilon_{\min,r}$, on the compact set $[0, r]$. Since $\epsilon > 0$ according to (A-6), it follows that $\epsilon_{\min,r} > 0$ for each $r \in \mathbb{R}^+$.

Using the presumed opaqueness together with Kirchhoff's law yields

$$J_r = (1 - \epsilon) J. \quad (5)$$

Due to diffuseness, the irradiation can be calculated as

$$J(\theta) = K(R(\theta)), \quad (6)$$

using the nonlocal integral radiation operator K defined by

$$K(\rho)(x) := \int_{\Sigma} \Lambda(x, y) \omega(x, y) \rho(y) dy \quad \text{for a.e. } x \in \Sigma, \quad (7)$$

$$\omega(x, y) := \frac{(\mathbf{n}_s(y) \cdot (x - y)) (\mathbf{n}_s(x) \cdot (y - x))}{\pi((y - x) \cdot (y - x))^2} \quad \text{for a.e. } (x, y) \in \Sigma \times \Sigma, \quad (8)$$

$$\Lambda(x, y) := \begin{cases} 0 & \text{if } \Sigma \cap]x, y[\neq \emptyset, \\ 1 & \text{if } \Sigma \cap]x, y[= \emptyset \end{cases} \quad \text{for each } (x, y) \in \Sigma \times \Sigma, \quad (9)$$

where ω is called view factor, Λ is called visibility factor (being 1 if, and only if, x and y are mutually visible), and \mathbf{n}_s denotes the outer unit normal to the solid domain Ω_s , existing almost everywhere on the Lipschitz interface Σ . The following Th. 2.1 summarizes properties of ω , Λ , and K , relevant to our considerations.

Theorem 2.1. Assume (A-1) and (A-2).

- (a) The kernel $\Lambda\omega$ of K is almost everywhere nonnegative (actually positive for $\Lambda(x, y) = 1$), symmetric, and $\Lambda(x, \cdot)\omega(x, \cdot)$ is in $L^1(\Sigma)$ with

$$\int_{\Sigma} \Lambda(x, y) \omega(x, y) dy = 1 \quad \text{for a.e. } x \in \Sigma. \quad (10)$$

- (b) For each $1 \leq p \leq \infty$, the operator $K : L^p(\Sigma) \rightarrow L^p(\Sigma)$ given by (7) is well-defined, linear, bounded, and positive with $\|K\| = 1$.

Proof. See [11, Lem. 1] and [12, Lem. 2]. \square

Combining (3) through (6) provides the so-called radiosity equation for R :

$$(\text{Id} - (1 - \epsilon(\theta))K)(R) = \sigma \epsilon(\theta) \theta^4, \quad (11)$$

where Id denotes the identity operator. The following Th. 2.2 allows to solve (11) for R .

Theorem 2.2. Assume (A-1), (A-2), (A-6).

- (a) Let $p \in [1, \infty]$. Then, for each $\theta \in L^1(\Sigma)$, the operator $\text{Id} - (1 - \epsilon(\theta))K$ has an inverse in the Banach space $\mathcal{L}(L^p(\Sigma), L^p(\Sigma))$ of bounded linear operators.
 (b) Let $p \in [1, \infty]$. For each $\theta \in L^{4p}(\Sigma)$, the radiosity equation (11) has the unique solution $R(\theta) = (\text{Id} - (1 - \epsilon(\theta))K)^{-1}(\sigma \epsilon(\theta) \theta^4) \in L^p(\Sigma)$.

Proof. Part (a) is given by [5, Th. 5], since Σ is assumed polyhedral by (A-1); (b) follows from (a), as $\sigma > 0$ and $\epsilon(\theta) \in L^\infty(\Sigma)$ by (A-6). \square

With the computation

$$\begin{aligned} R(\theta) - J(\theta) &\stackrel{(3)}{=} E(\theta) + J_r(\theta) - J(\theta) \stackrel{(4),(5)}{=} \sigma \epsilon(\theta) \theta^4 - \epsilon(\theta) J(\theta) \\ &\stackrel{(6)}{=} -\epsilon(\theta) (K(R(\theta)) - \sigma \theta^4), \end{aligned} \quad (12)$$

(2) becomes

$$(\kappa_g \nabla \theta) \upharpoonright_{\overline{\Omega}_g} \cdot \mathbf{n}_g - \epsilon(\theta) (K(R(\theta)) - \sigma \theta^4) = (\kappa_s \nabla \theta) \upharpoonright_{\overline{\Omega}_s} \cdot \mathbf{n}_g \quad \text{on }]0, T[\times \Sigma, \quad (13)$$

where $R(\theta)$ is given by Th. 2.2(b).

2.3. Nonlocal Outer Boundary Conditions.

Definition 2.3. Let $\text{conv}(\overline{\Omega})$ denote the closed convex hull of Ω , and define $O := \text{int}(\text{conv}(\overline{\Omega})) \setminus \overline{\Omega}$, $\Gamma_\Omega := \overline{\Omega} \cap \overline{O}$, $\Gamma := \partial O$, and $\Gamma_{\text{ph}} := \partial \text{conv}(\overline{\Omega}) \cap \partial O$. Then $(\Gamma_\Omega, \Gamma_{\text{ph}})$ forms a partition of Γ . The set O is the domain of the open radiation region (e.g., one has $O = O_1 \cup O_2$ in Figures 1 and 2).

The condition on the interface between Ω and the open radiation region O reads

$$\kappa_s \nabla \theta \cdot \mathbf{n}_s + R_\Gamma(\theta) - J_\Gamma(\theta) = 0 \quad \text{on }]0, T[\times \Gamma_\Omega \quad (14)$$

in analogy with (2), where \mathbf{n}_s is the outer unit normal vector to the solid. To allow for radiative interactions between surfaces of open cavities and the ambient environment, including reflections at the cavity's surfaces, the set Γ_{ph} as defined above, is used as a black body phantom closure (see Fig. 2), emitting radiation at an external temperature θ_{ext} ,

(A-7) $\theta_{\text{ext}} \in \mathbb{R}^+$.

$$R_{\Gamma}(\theta)(x) = \sigma \theta_{\text{ext}}^4 \quad (x \in \Gamma_{\text{ph}}). \quad (15)$$
$$\kappa_s \nabla \theta \cdot \mathbf{n}_s - \epsilon(\theta) (K_\Gamma(R_\Gamma(\theta)) - \sigma \theta^4) = 0 \quad \text{on }]0, T[\times \Gamma_\Omega, \quad (16)$$

Diagram illustrating the decomposition of a set O into three regions $\Omega_{g,1}$, $\Omega_{g,2}$, and $\Omega_{g,3}$, and their relationship to the convex hull of \overline{O} .

The diagram shows a complex shape \overline{O} (shaded gray) with a hole O (white). The boundary of \overline{O} is dashed, and the boundary of O is solid. The regions $\Omega_{g,1}$, $\Omega_{g,2}$, and $\Omega_{g,3}$ are defined as the interior of the convex hull of \overline{O} minus O .

The diagram also shows the decomposition of O into three regions O_1 , O_2 , and O_3 . The boundary of O is solid, and the boundary of \overline{O} is dashed. The regions $\Omega_{g,1}$, $\Omega_{g,2}$, and $\Omega_{g,3}$ are defined as the interior of the convex hull of \overline{O} minus O .

Mathematical definitions shown in the diagram:

- $\overline{O} = \overline{\Omega_s} \cup \overline{\Omega_g}$
- $\partial \Omega_g$ (dashed line)
- $\Sigma = \partial \Omega_g = \overline{\Omega_s} \cap \overline{\Omega_g}$
- $O := \text{int}(\text{conv}(\overline{O})) \setminus \overline{O}$
- $O = O_1 \cup O_2$
- $\Gamma_O := \overline{O} \cap \overline{O}$ (solid line)
- $\Gamma_{ph} := \partial \text{conv}(\overline{O}) \cap \partial O$ (green line)
- $\Gamma := \partial O = \Gamma_O \cup \Gamma_{ph}$

$$\kappa_s \nabla \theta \cdot \mathbf{n}_s - \sigma \epsilon(\theta) (\theta_{\text{ext}}^4 - \theta^4) = 0 \quad \text{on }]0, T[\times (\partial\Omega \setminus \Gamma_\Omega). \quad (17)$$

(A-8) $\theta_{\text{init}} \in L^\infty(\Omega, \mathbb{R}^+)$.

An admissible discretization of the space domain Ω is given by a finite family $\mathcal{T} := (\omega_i)_{i \in I}$ of subsets of Ω satisfying a number of assumptions, subsequently denoted by (DA-*).

(DA-1) $\mathcal{T} = (\omega_i)_{i \in I}$ forms a partition of Ω , and, for each $i \in I$, ω_i is a nonvoid, polyhedral, connected, and open subset of Ω .

From \mathcal{T} , one can define discretizations of Ω_s and Ω_g : For $m \in \{s, g\}$ and $i \in I$, let

$$\omega_{m,i} := \omega_i \cap \Omega_m, \quad I_m := \{j \in I : \omega_{m,j} \neq \emptyset\}, \quad \mathcal{T}_m := (\omega_{m,i})_{i \in I_m}. \quad (18)$$

(DA-2) For each $i \in I$: $\partial_{\text{reg}} \omega_{s,i} \cap \Sigma = \partial_{\text{reg}} \omega_{g,i} \cap \Sigma$, where ∂_{reg} denotes the regular boundary of a polyhedral set, i.e. the parts of the boundary, where a unique outer unit normal vector exists, $\partial_{\text{reg}} \emptyset := \emptyset$.

The boundary of each control volume $\omega_{m,i}$ can be decomposed according to

$$\partial \omega_{m,i} = (\partial \omega_{m,i} \cap \Omega_m) \cup (\partial \omega_{m,i} \cap \partial \Omega) \cup (\partial \omega_{m,i} \cap \Sigma). \quad (19a)$$

Recalling (A-1), (A-2), and Def. 2.3, outer boundary sets are decomposed further into

$$\partial \omega_{s,i} \cap \partial \Omega = (\partial \omega_{s,i} \cap \Gamma_\Omega) \cup (\partial \omega_{s,i} \cap (\partial \Omega \setminus \Gamma_\Omega)), \quad (19b)$$

whereas $\partial \omega_{g,i} \cap \partial \Omega = \emptyset$.

Associate a discretization point $x_i \in \overline{\omega_i}$ with each control volume ω_i (cf. [7]). Then $\theta_{\nu,i}$ can be interpreted as $\theta(t_\nu, x_i)$. Moreover, the discretization makes use of regularity assumptions concerning the partition $(\omega_i)_{i \in I}$ that can be expressed in terms of the x_i :

(DA-3) For each $m \in \{s, g\}$, $i \in I_m$: $x_i \in \overline{\omega_{m,i}}$. In particular, if $\omega_{s,i} \neq \emptyset$ and $\omega_{g,i} \neq \emptyset$, then $x_i \in \overline{\omega_{s,i}} \cap \overline{\omega_{g,i}}$.

(DA-4) For each $i \in I$, the following holds: If $\lambda_2(\overline{\omega_i} \cap \Gamma_\Omega) \neq 0$, then $x_i \in \overline{\omega_i} \cap \Gamma_\Omega$; and, if $\lambda_2(\overline{\omega_i} \cap (\partial \Omega \setminus \Gamma_\Omega)) \neq 0$, then $x_i \in \overline{\omega_i} \cap \partial \Omega \setminus \Gamma_\Omega$.

Remark 3. By (A-2), (DA-2), (DA-3), (DA-4), $\overline{\omega_i}$ can *not* have 2-dimensional intersections with both $\partial \Omega$ and Σ .

Introducing the sets

$$\text{nb}_m(i) := \{j \in I_m \setminus \{i\} : \lambda_2(\partial \omega_{m,i} \cap \partial \omega_{m,j}) \neq 0\}, \quad (20a)$$

$$\text{nb}(i) := \{j \in I \setminus \{i\} : \lambda_2(\partial \omega_i \cap \partial \omega_j) \neq 0\}, \quad (20b)$$

$\partial \omega_{m,i} \cap \Omega_m$ is partitioned further:

$$\partial \omega_{m,i} \cap \Omega_m = \bigcup_{j \in \text{nb}_m(i)} \partial \omega_{m,i} \cap \partial \omega_{m,j}, \quad (21)$$

where it is assumed that:

(DA-5) For each $i \in I$, $j \in \text{nb}(i)$: $x_i \neq x_j$ and $\frac{x_j - x_i}{\|x_i - x_j\|_2} = \mathbf{n}_{\omega_i} \upharpoonright_{\partial \omega_i \cap \partial \omega_j}$, where $\|\cdot\|_2$ denotes Euclidean distance, and $\mathbf{n}_{\omega_i} \upharpoonright_{\partial \omega_i \cap \partial \omega_j}$ is the restriction of the normal vector \mathbf{n}_{ω_i} to the interface $\partial \omega_i \cap \partial \omega_j$. Thus, the line segment joining neighboring vertices x_i and x_j is always perpendicular to $\partial \omega_i \cap \partial \omega_j$.

3.2. Discretization of Nonlocal Radiation Terms.

(DA-5) For a chosen fixed index “ph”, $(\zeta_\alpha)_{\alpha \in I_\Omega}$ and $(\zeta_\alpha)_{\alpha \in I_\Sigma}$ are finite partitions of Γ_Ω and Σ , respectively, where

$$I_\Omega \cap I_\Sigma = \emptyset, \quad \text{ph} \notin I_\Omega \cup I_\Sigma, \quad (22)$$

and, for each $\alpha \in I_\Omega$ (resp. $\alpha \in I_\Sigma$), the boundary element ζ_α is a nonvoid, polyhedral, connected, and (relatively) open subset of Γ_Ω (resp. Σ), lying in a 2-dimensional affine subspace of \mathbb{R}^3 . For the convenience of subsequent concise notation, let $\zeta_{\text{ph}} := \Gamma_{\text{ph}}$ and $I_\Gamma := I_\Omega \dot{\cup} \{\text{ph}\}$.

On both Γ_Ω and Σ , the boundary elements are supposed to be compatible with the control volumes ω_i :

(DA-6) For each $\alpha \in I_\Omega$ (resp. $\alpha \in I_\Sigma$), there is a unique $i(\alpha) \in I$ such that $\zeta_\alpha \subseteq \partial\omega_{i(\alpha)} \cap \Gamma_\Omega$ (resp. $\zeta_\alpha \subseteq \partial\omega_{s,i(\alpha)} \cap \Gamma_\Sigma$). Moreover, for each $\alpha \in I_\Omega \dot{\cup} I_\Sigma$: $x_{i(\alpha)} \in \bar{\zeta}_\alpha$.

For each $i \in I$, define $J_{\Omega,i} := \{\alpha \in I_\Omega : \lambda_2(\zeta_\alpha \cap \partial\omega_i) \neq 0\}$ and $J_{\Sigma,i} := \{\alpha \in I_\Sigma : \lambda_2(\zeta_\alpha \cap \partial\omega_{s,i}) \neq 0\}$.

Remark 4. As a consequence of (DA-1), (DA-5), and (DA-6), the family $(\zeta_\alpha \cap \partial\omega_i)_{\alpha \in J_{\Omega,i}}$ is a partition of $\partial\omega_i \cap \Gamma_\Omega = \partial\omega_{s,i} \cap \Gamma_\Omega$ and $(\zeta_\alpha \cap \partial\omega_{s,i})_{\alpha \in J_{\Sigma,i}}$ is a partition of $\partial\omega_{s,i} \cap \Sigma = \bar{\omega}_i \cap \Sigma$. Moreover, (A-2) implies that at most one of the two sets $J_{\Omega,i}$, $J_{\Sigma,i}$ can be nonvoid (cf. Rem. 3 above).

Let

$$\Lambda_{\alpha,\beta} := \int_{\zeta_\alpha \times \zeta_\beta} \Lambda \omega \quad \text{for all } (\alpha, \beta) \in (I_\Sigma \times I_\Sigma) \cup (I_\Gamma \times I_\Gamma). \quad (23)$$

The $\Lambda_{\alpha,\beta}$ are nonnegative since $\Lambda \omega$ is nonnegative according to Th. 2.1(a). The forms of Λ and ω imply the symmetry condition $\Lambda_{\alpha,\beta} = \Lambda_{\beta,\alpha}$; and (10) (resp. its analog, where Σ is replaced by $\Gamma = \Gamma_\Omega \cup \Gamma_{\text{ph}}$) implies

$$\sum_{\beta \in I_\Sigma} \Lambda_{\alpha,\beta} = \lambda_2(\zeta_\alpha) \quad \text{for all } \alpha \in I_\Sigma, \quad \sum_{\beta \in I_\Gamma} \Lambda_{\alpha,\beta} = \lambda_2(\zeta_\alpha) \quad \text{for all } \alpha \in I_\Gamma. \quad (24)$$

Define vector-valued functions

$$\begin{aligned} \mathbf{E}_\Gamma : (\mathbb{R}_0^+)^{I_\Omega} &\longrightarrow (\mathbb{R}_0^+)^{I_\Omega}, & \mathbf{E}_\Gamma(\mathbf{u}) &= (E_{\Gamma,\alpha}(\mathbf{u}))_{\alpha \in I_\Omega}, \\ \mathbf{E}_\Sigma : (\mathbb{R}_0^+)^{I_\Sigma} &\longrightarrow (\mathbb{R}_0^+)^{I_\Sigma}, & \mathbf{E}_\Sigma(\mathbf{u}) &= (E_{\Sigma,\alpha}(\mathbf{u}))_{\alpha \in I_\Sigma}, \\ E_{\Gamma,\alpha}(\mathbf{u}) &:= E_{\Sigma,\alpha}(\mathbf{u}) := \sigma \epsilon(u_\alpha) u_\alpha^4 \lambda_2(\zeta_\alpha), \end{aligned} \quad (25a)$$

$$\begin{aligned} \mathbf{E}_{\text{ph}} : (\mathbb{R}_0^+)^{I_\Omega} &\longrightarrow (\mathbb{R}_0^+)^{I_\Omega}, & \mathbf{E}_{\text{ph}}(\mathbf{u}) &= (E_{\text{ph},\alpha}(\mathbf{u}))_{\alpha \in I_\Omega}, \\ E_{\text{ph},\alpha}(\mathbf{u}) &:= \sigma (1 - \epsilon(u_\alpha)) \theta_{\text{ext}}^4 \Lambda_{\alpha,\text{ph}}, \end{aligned} \quad (25b)$$

and matrix-valued functions

$$\begin{aligned} \mathbf{G}_\Gamma : (\mathbb{R}_0^+)^{I_\Omega} &\longrightarrow \mathbb{R}^{I_\Omega^2}, & \mathbf{G}_\Gamma(\mathbf{u}) &= (G_{\Gamma,\alpha,\beta}(\mathbf{u}))_{(\alpha,\beta) \in I_\Omega^2}, \\ \mathbf{G}_\Sigma : (\mathbb{R}_0^+)^{I_\Sigma} &\longrightarrow \mathbb{R}^{I_\Sigma^2}, & \mathbf{G}_\Sigma(\mathbf{u}) &= (G_{\Sigma,\alpha,\beta}(\mathbf{u}))_{(\alpha,\beta) \in I_\Sigma^2}, \\ G_{\Gamma,\alpha,\beta}(\mathbf{u}) &:= G_{\Sigma,\alpha,\beta}(\mathbf{u}) := \begin{cases} \lambda_2(\zeta_\alpha) - (1 - \epsilon(u_\alpha)) \Lambda_{\alpha,\beta} & \text{for } \alpha = \beta, \\ - (1 - \epsilon(u_\alpha)) \Lambda_{\alpha,\beta} & \text{for } \alpha \neq \beta. \end{cases} \end{aligned} \quad (25c)$$

Lemma 3.1. *The following holds for each $\mathbf{u} \in (\mathbb{R}_0^+)^{I_\Omega}$:*

- (a) *For each $\alpha \in I_\Omega$: $\sum_{\beta \in I_\Omega \setminus \{\alpha\}} |G_{\Gamma,\alpha,\beta}(\mathbf{u})| \leq (1 - \epsilon(u_\alpha)) G_{\Gamma,\alpha,\alpha}(\mathbf{u}) < G_{\Gamma,\alpha,\alpha}(\mathbf{u})$. In particular, $\mathbf{G}_\Gamma(\mathbf{u})$ is strictly diagonally dominant.*
- (b) *$\mathbf{G}_\Gamma(\mathbf{u})$ is an M-matrix, i.e. $\mathbf{G}_\Gamma(\mathbf{u})$ is invertible, $\mathbf{G}_\Gamma^{-1}(\mathbf{u})$ is nonnegative, and, for each $(\alpha, \beta) \in I_\Omega^2$ such that $\alpha \neq \beta$: $G_{\Gamma,\alpha,\beta}(\mathbf{u}) \leq 0$.*

Analogous statements hold for \mathbf{G}_Σ .

Proof. The proof is completely analogous to the proof of [9, Lem. 3.1]. □

Using Lem. 3.1, one can define the vector-valued functions

$$\mathbf{R}_\Gamma : (\mathbb{R}_0^+)^{I_\Omega} \longrightarrow (\mathbb{R}_0^+)^{I_\Omega}, \quad \mathbf{R}_\Gamma(\mathbf{u}) := \mathbf{G}_\Gamma^{-1}(\mathbf{u}) (\mathbf{E}_\Gamma(\mathbf{u}) + \mathbf{E}_{\text{ph}}(\mathbf{u})), \quad (26a)$$

$$\mathbf{R}_\Sigma : (\mathbb{R}_0^+)^{I_\Sigma} \longrightarrow (\mathbb{R}_0^+)^{I_\Sigma}, \quad \mathbf{R}_\Sigma(\mathbf{u}) := \mathbf{G}_\Sigma^{-1}(\mathbf{u}) \mathbf{E}_\Sigma(\mathbf{u}), \quad (26b)$$

$$\mathbf{V}_\Gamma : (\mathbb{R}_0^+)^{I_\Omega} \longrightarrow (\mathbb{R}_0^+)^{I_\Omega}, \quad \mathbf{V}_\Gamma(\mathbf{u}) = (V_{\Gamma,\alpha}(\mathbf{u}))_{\alpha \in I_\Omega},$$

$$V_{\Gamma,\alpha}(\mathbf{u}) := \epsilon(u_\alpha) \sum_{\beta \in I_\Omega} R_{\Gamma,\beta}(\mathbf{u}) \Lambda_{\alpha,\beta} + \sigma \epsilon(u_\alpha) \theta_{\text{ext}}^4 \Lambda_{\alpha,\text{ph}}, \quad (26c)$$

$$\mathbf{V}_\Sigma : (\mathbb{R}_0^+)^{I_\Sigma} \longrightarrow (\mathbb{R}_0^+)^{I_\Sigma}, \quad \mathbf{V}_\Sigma(\mathbf{u}) = (V_{\Sigma,\alpha}(\mathbf{u}))_{\alpha \in I_\Sigma},$$

$$V_{\Sigma,\alpha}(\mathbf{u}) := \epsilon(u_\alpha) \sum_{\beta \in I_\Sigma} R_{\Sigma,\beta}(\mathbf{u}) \Lambda_{\alpha,\beta}. \quad (26d)$$

The following Lem. 3.2 corresponds to [9, Lem. 3.2] and provides a maximum principle as well as local Lipschitzness for the functions \mathbf{R}_Γ , \mathbf{V}_Γ , \mathbf{R}_Σ , and \mathbf{V}_Σ . Only part (c) of Lem. 3.2 is significantly different from its counterpart in [9, Lem. 3.2]. The following notation is introduced for $\mathbf{u} = (u_i)_{i \in I} \in \mathbb{R}^I$ (where I can be an arbitrary, nonempty, finite index set):

$$\|\mathbf{u}\|_{\min} := \min\{u_i : i \in I\}, \quad \|\mathbf{u}\|_{\max} := \max\{u_i : i \in I\}. \quad (27)$$

Lemma 3.2. (a) *The functions \mathbf{R}_Γ , \mathbf{V}_Γ , \mathbf{R}_Σ , and \mathbf{V}_Σ are all nonnegative.*

(b) *For each $\mathbf{u} \in (\mathbb{R}_0^+)^{I_\Omega}$, $\alpha \in I_\Omega$:*

$$\sigma \min\{\|\mathbf{u}\|_{\min}^4, \theta_{\text{ext}}^4\} \leq R_{\Gamma,\alpha}(\mathbf{u}) \leq \sigma \max\{\|\mathbf{u}\|_{\max}^4, \theta_{\text{ext}}^4\}, \quad (28a)$$

$$\begin{aligned} \sigma \epsilon(u_\alpha) \min\{\|\mathbf{u}\|_{\min}^4, \theta_{\text{ext}}^4\} \lambda_2(\zeta_\alpha) &\leq V_{\Gamma,\alpha}(\mathbf{u}) \\ &\leq \sigma \epsilon(u_\alpha) \max\{\|\mathbf{u}\|_{\max}^4, \theta_{\text{ext}}^4\} \lambda_2(\zeta_\alpha), \end{aligned} \quad (28b)$$

and, for each $\mathbf{u} \in (\mathbb{R}_0^+)^{I_\Sigma}$, $\alpha \in I_\Sigma$:

$$\sigma \|\mathbf{u}\|_{\min}^4 \leq R_{\Sigma,\alpha}(\mathbf{u}) \leq \sigma \|\mathbf{u}\|_{\max}^4, \quad (28c)$$

$$\sigma \epsilon(u_\alpha) \|\mathbf{u}\|_{\min}^4 \lambda_2(\zeta_\alpha) \leq V_{\Sigma,\alpha}(\mathbf{u}) \leq \sigma \epsilon(u_\alpha) \|\mathbf{u}\|_{\max}^4 \lambda_2(\zeta_\alpha). \quad (28d)$$

(c) *For each $r \in \mathbb{R}^+$, with respect to the max-norm, the maps $R_{\Gamma,\alpha}$, $V_{\Gamma,\alpha}$ are Lipschitz on $[0, r]^{I_\Omega}$, and the maps $R_{\Sigma,\alpha}$, and $V_{\Sigma,\alpha}$ are Lipschitz on $[0, r]^{I_\Sigma}$. More precisely, recalling $\epsilon_{\min,r} \in \mathbb{R}^+$ from Rem. 2 and $L_{\epsilon,r}$ from (A-6), the Lipschitz constants are*

$$\sigma \epsilon_{\min,r}^{-1} \left(4r^3 + L_{\epsilon,r} (r^4 + \max\{r^4, \theta_{\text{ext}}^4\}) \right) \quad \text{for } \mathbf{R}_\Gamma, \quad (29a)$$

$$\begin{aligned} \sigma \lambda_2(\zeta_\alpha) \left(4\epsilon(u_\alpha) \epsilon_{\min,r}^{-1} r^3 \right. \\ \left. + \max\{r^4, \theta_{\text{ext}}^4\} L_{\epsilon,r} (2\epsilon(u_\alpha) \epsilon_{\min,r}^{-1} + 1) \right) \quad \text{for } V_{\Gamma,\alpha}, \end{aligned} \quad (29b)$$

$$\sigma \epsilon_{\min,r}^{-1} (4r^3 + 2L_{\epsilon,r} r^4) \quad \text{for } \mathbf{R}_\Sigma, \quad (29c)$$

$$\sigma \lambda_2(\zeta_\alpha) \left(4\epsilon(u_\alpha) \epsilon_{\min,r}^{-1} r^3 + r^4 L_{\epsilon,r} (2\epsilon(u_\alpha) \epsilon_{\min,r}^{-1} + 1) \right) \quad \text{for } V_{\Sigma,\alpha}. \quad (29d)$$

Proof. The proofs of (a) and (b) are completely analogous to the proofs of [9, Lem. 3.2(a),(b)].

(c): Note that (26a) implies

$$\mathbf{G}_\Gamma(\mathbf{u}) \mathbf{R}_\Gamma(\mathbf{u}) = \mathbf{E}_\Gamma(\mathbf{u}) + \mathbf{E}_{\text{ph}}(\mathbf{u}), \quad (30)$$

or, written in components:

$$\begin{aligned} R_{\Gamma,\alpha}(\mathbf{u}) \lambda_2(\zeta_\alpha) - (1 - \epsilon(u_\alpha)) \sum_{\beta \in I_\Omega} R_{\Gamma,\beta}(\mathbf{u}) \Lambda_{\alpha,\beta} \\ = \sigma \epsilon(u_\alpha) u_\alpha^4 \lambda_2(\zeta_\alpha) + \sigma (1 - \epsilon(u_\alpha)) \theta_{\text{ext}}^4 \Lambda_{\alpha,\text{ph}} \end{aligned} \quad (\alpha \in I_\Omega). \quad (31)$$

The function $\theta \mapsto \theta^4$ is $(4r^3)$ -Lipschitz on $[0, r]$, such that, by (31), for each $(\mathbf{u}, \mathbf{v}) \in [0, r]^{I_\Omega} \times [0, r]^{I_\Omega}$, $\alpha \in I_\Omega$:

$$\begin{aligned} & \left| (R_{\Gamma,\alpha}(\mathbf{u}) - R_{\Gamma,\alpha}(\mathbf{v})) \lambda_2(\zeta_\alpha) \right. \\ & \quad - \sum_{\beta \in I_\Omega} \left((1 - \epsilon(u_\alpha)) R_{\Gamma,\beta}(\mathbf{u}) - (1 - \epsilon(v_\alpha)) R_{\Gamma,\beta}(\mathbf{v}) \right) \Lambda_{\alpha,\beta} \\ & \quad \left. - \sigma \left((1 - \epsilon(u_\alpha)) - (1 - \epsilon(v_\alpha)) \right) \theta_{\text{ext}}^4 \Lambda_{\alpha,\text{ph}} \right| \\ & = \sigma \left| \epsilon(u_\alpha) u_\alpha^4 - \epsilon(v_\alpha) v_\alpha^4 \right| \lambda_2(\zeta_\alpha) \stackrel{(\text{A-6})}{\leq} \sigma (4r^3 + L_{\epsilon,r} r^4) |u_\alpha - v_\alpha| \lambda_2(\zeta_\alpha). \end{aligned} \quad (32)$$

Now, let $\alpha \in I_\Omega$ be such that $N_{\max} := \|\mathbf{R}_\Gamma(\mathbf{u}) - \mathbf{R}_\Gamma(\mathbf{v})\|_{\max} = |R_{\Gamma,\alpha}(\mathbf{u}) - R_{\Gamma,\alpha}(\mathbf{v})|$. Then one can estimate

$$\begin{aligned} & \left| \sigma (\epsilon(v_\alpha) - \epsilon(u_\alpha)) \theta_{\text{ext}}^4 \Lambda_{\alpha,\text{ph}} \right. \\ & \quad \left. + \sum_{\beta \in I_\Omega} \left((1 - \epsilon(u_\alpha)) R_{\Gamma,\beta}(\mathbf{u}) - (1 - \epsilon(v_\alpha)) R_{\Gamma,\beta}(\mathbf{v}) \right) \Lambda_{\alpha,\beta} \right| \\ & \leq \sigma L_{\epsilon,r} \|\mathbf{u} - \mathbf{v}\|_{\max} \theta_{\text{ext}}^4 \Lambda_{\alpha,\text{ph}} \\ & \quad + \left| \sum_{\beta \in I_\Omega} \left((1 - \epsilon(u_\alpha)) R_{\Gamma,\beta}(\mathbf{u}) - (1 - \epsilon(u_\alpha)) R_{\Gamma,\beta}(\mathbf{v}) \right) \Lambda_{\alpha,\beta} \right| \\ & \quad + \left| \sum_{\beta \in I_\Omega} \left((1 - \epsilon(u_\alpha)) R_{\Gamma,\beta}(\mathbf{v}) - (1 - \epsilon(v_\alpha)) R_{\Gamma,\beta}(\mathbf{v}) \right) \Lambda_{\alpha,\beta} \right| \\ & \leq \sigma L_{\epsilon,r} \|\mathbf{u} - \mathbf{v}\|_{\max} \theta_{\text{ext}}^4 \Lambda_{\alpha,\text{ph}} + (1 - \epsilon(u_\alpha)) N_{\max} (\lambda_2(\zeta_\alpha) - \Lambda_{\alpha,\text{ph}}) \\ & \quad + \sigma \max\{r^4, \theta_{\text{ext}}^4\} L_{\epsilon,r} \|\mathbf{u} - \mathbf{v}\|_{\max} (\lambda_2(\zeta_\alpha) - \Lambda_{\alpha,\text{ph}}) \\ & \leq (1 - \epsilon(u_\alpha)) N_{\max} \lambda_2(\zeta_\alpha) + \sigma \max\{r^4, \theta_{\text{ext}}^4\} L_{\epsilon,r} \|\mathbf{u} - \mathbf{v}\|_{\max} \lambda_2(\zeta_\alpha). \end{aligned} \quad (33)$$

Next, (32) and (33) imply

$$\begin{aligned} & \sigma (4r^3 + L_{\epsilon,r} r^4) \|\mathbf{u} - \mathbf{v}\|_{\max} \lambda_2(\zeta_\alpha) \\ & \stackrel{(32)}{\geq} \left| N_{\max} \lambda_2(\zeta_\alpha) - \sum_{\beta \in I_\Omega} \left((1 - \epsilon(u_\alpha)) R_{\Gamma,\beta}(\mathbf{u}) - (1 - \epsilon(v_\alpha)) R_{\Gamma,\beta}(\mathbf{v}) \right) \Lambda_{\alpha,\beta} \right. \\ & \quad \left. + \sigma (\epsilon(v_\alpha) - \epsilon(u_\alpha)) \theta_{\text{ext}}^4 \Lambda_{\alpha,\text{ph}} \right| \end{aligned}$$

$$\begin{aligned}
 &\geq N_{\max} \lambda_2(\zeta_\alpha) - \left| \sum_{\beta \in I_\Omega} \left((1 - \epsilon(u_\alpha)) R_{\Gamma, \beta}(\mathbf{u}) - (1 - \epsilon(v_\alpha)) R_{\Gamma, \beta}(\mathbf{v}) \right) \Lambda_{\alpha, \beta} \right. \\
 &\quad \left. + \sigma (\epsilon(v_\alpha) - \epsilon(u_\alpha)) \theta_{\text{ext}}^4 \Lambda_{\alpha, \text{ph}} \right| \\
 &\stackrel{(33)}{\geq} N_{\max} \left(\lambda_2(\zeta_\alpha) - (1 - \epsilon(u_\alpha)) \lambda_2(\zeta_\alpha) \right) \\
 &\quad - \sigma \max\{r^4, \theta_{\text{ext}}^4\} L_{\epsilon, r} \|\mathbf{u} - \mathbf{v}\|_{\max} \lambda_2(\zeta_\alpha) \\
 &= N_{\max} \epsilon(u_\alpha) \lambda_2(\zeta_\alpha) - \sigma \max\{r^4, \theta_{\text{ext}}^4\} L_{\epsilon, r} \|\mathbf{u} - \mathbf{v}\|_{\max} \lambda_2(\zeta_\alpha),
 \end{aligned}$$

thereby proving (29a).

To prove the claimed Lipschitz continuity (29b) of $\mathbf{V}_{\Gamma, \alpha}$, $\alpha \in I_\Omega$, one uses (26c) and estimates

$$\begin{aligned}
 &|V_{\Gamma, \alpha}(\mathbf{u}) - V_{\Gamma, \alpha}(\mathbf{v})| \\
 &\stackrel{(26c), (A-6)}{\leq} \left| \sum_{\beta \in I_\Omega} (\epsilon(u_\alpha) R_{\Gamma, \beta}(\mathbf{u}) - \epsilon(u_\alpha) R_{\Gamma, \beta}(\mathbf{v})) \Lambda_{\alpha, \beta} \right| \\
 &\quad + \left| \sum_{\beta \in I_\Omega} (\epsilon(u_\alpha) R_{\Gamma, \beta}(\mathbf{v}) - \epsilon(v_\alpha) R_{\Gamma, \beta}(\mathbf{v})) \Lambda_{\alpha, \beta} \right| \\
 &\quad + \sigma L_{\epsilon, r} |u_\alpha - v_\alpha| \theta_{\text{ext}}^4 \Lambda_{\alpha, \text{ph}} \\
 &\stackrel{(29a), (A-6)}{\leq} \epsilon(u_\alpha) \sigma \epsilon_{\min, r}^{-1} \left(4 r^3 + L_{\epsilon, r} (r^4 + \max\{r^4, \theta_{\text{ext}}^4\}) \right) \\
 &\quad \cdot \|\mathbf{u} - \mathbf{v}\|_{\max} (\lambda_2(\zeta_\alpha) - \Lambda_{\alpha, \text{ph}}) \\
 &\quad + \sigma \max\{r^4, \theta_{\text{ext}}^4\} L_{\epsilon, r} \|\mathbf{u} - \mathbf{v}\|_{\max} (\lambda_2(\zeta_\alpha) - \Lambda_{\alpha, \text{ph}}) \\
 &\quad + \sigma L_{\epsilon, r} \|\mathbf{u} - \mathbf{v}\|_{\max} \theta_{\text{ext}}^4 \Lambda_{\alpha, \text{ph}} \\
 &\leq \sigma \max\{r^4, \theta_{\text{ext}}^4\} L_{\epsilon, r} \lambda_2(\zeta_\alpha) (2 \epsilon(u_\alpha) \epsilon_{\min, r}^{-1} + 1) \|\mathbf{u} - \mathbf{v}\|_{\max} \\
 &\quad + 4 \epsilon(u_\alpha) \sigma \epsilon_{\min, r}^{-1} r^3 \lambda_2(\zeta_\alpha) \|\mathbf{u} - \mathbf{v}\|_{\max},
 \end{aligned}$$

which establishes (29b).

The assertions (29c) and (29d) on \mathbf{R}_Σ and $V_{\Sigma, \alpha}$, respectively, can be proved analogously to the proofs of (29a) and (29b) above. \square

3.3. Formulation of Scheme. Recalling the meaning of $i(\alpha)$ from (DA-6), for each $\mathbf{u} = (u_i)_{i \in I}$, define

$$\mathbf{u}|_{I_\Omega} := (u_{i(\alpha)})_{\alpha \in I_\Omega}, \quad \mathbf{u}|_{I_\Sigma} := (u_{i(\alpha)})_{\alpha \in I_\Sigma}. \quad (34)$$

The finite volume scheme is now stated in (35) and (36) below. One is seeking a nonnegative solution $(\mathbf{u}_0, \dots, \mathbf{u}_N)$, $\mathbf{u}_\nu = (u_{\nu, i})_{i \in I}$, to

$$u_{0, i} = \theta_{\text{init}, i} \quad (i \in I), \quad (35a)$$

$$\mathcal{H}_{\nu, i}(\mathbf{u}_{\nu-1}, \mathbf{u}_\nu) = 0 \quad (i \in I, \quad \nu \in \{1, \dots, N\}), \quad (35b)$$

where, for each $\nu \in \{1, \dots, N\}$:

$$\mathcal{H}_{\nu, i} : (\mathbb{R}_0^+)^I \times (\mathbb{R}_0^+)^I \longrightarrow \mathbb{R},$$

$$\mathcal{H}_{\nu,i}(\tilde{\mathbf{u}}, \mathbf{u}) = k_\nu^{-1} \sum_{m \in \{s,g\}} (\varepsilon_m(u_i) - \varepsilon_m(\tilde{u}_i)) \lambda_3(\omega_{m,i}) \quad (36a)$$

$$- \sum_{m \in \{s,g\}} \kappa_m \sum_{j \in \text{nb}_m(i)} \frac{u_j - u_i}{\|x_i - x_j\|_2} \lambda_2(\partial\omega_{m,i} \cap \partial\omega_{m,j}) \quad (36b)$$

$$+ \sigma \varepsilon(u_i) u_i^4 \lambda_2(\partial\omega_{s,i} \cap \Gamma_\Omega) - \sum_{\alpha \in J_{\Omega,i}} V_{\Gamma,\alpha}(\mathbf{u}|_{I_\Omega}) \quad (36c)$$

$$+ \sigma \varepsilon(u_i) (u_i^4 - \theta_{\text{ext}}^4) \lambda_2(\partial\omega_{s,i} \cap (\partial\Omega \setminus \Gamma_\Omega)) \quad (36d)$$

$$+ \sigma \varepsilon(u_i) u_i^4 \lambda_2(\omega_i \cap \Sigma) - \sum_{\alpha \in J_{\Sigma,i}} V_{\Sigma,\alpha}(\mathbf{u}|_{I_\Sigma}) \quad (36e)$$

$$- \sum_{m \in \{s,g\}} f_{m,\nu,i} \lambda_3(\omega_{m,i}), \quad (36f)$$

where

$$f_{m,\nu,i} \approx \frac{\int_{t_{\nu-1}}^{t_\nu} \int_{\omega_{m,i}} f_m}{k_\nu \lambda_3(\omega_{m,i})} \quad (37)$$

is a suitable approximation of the source term on $]t_{\nu-1}, t_\nu[\times \omega_{m,i}$, and $\theta_{\text{init},i}$ is a suitable approximation of θ_{init} on ω_i , $i \in I$. In general, the choices will depend on the regularity of f_m and θ_{init} (for f_m continuous, one might choose $f_{m,\nu,i} := f_m(t_\nu, x_i)$, but $f_{m,\nu,i} := (k_\nu \lambda_3(\omega_{m,i}))^{-1} \int_{t_{\nu-1}}^{t_\nu} \int_{\omega_{m,i}} f_m$ for a general $f_m \in L^\infty([0, T] \times \Omega_m)$). However, suitable approximations are assumed to satisfy:

(AA-1) For each $m \in \{s, g\}$, $\nu \in \{0, \dots, N\}$, and $i \in I$:

$$0 \leq \text{ess inf}(f_m|_{]t_{\nu-1}, t_\nu[\times \omega_{m,i}}) \leq f_{m,\nu,i} \leq \|f_m\|_{L^\infty([t_{\nu-1}, t_\nu] \times \omega_{m,i})},$$

where ess inf denotes the essential infimum.

(AA-2) For each $i \in I$:

$$0 \leq \text{ess inf}(\theta_{\text{init}}|_{\omega_i}) \leq \theta_{\text{init},i} \leq \|\theta_{\text{init}}\|_{L^\infty(\omega_i)}.$$

Remark 5. As in [9, Sec. 3.6], one can consider the case where Ω_s and Ω_g are axisymmetric, and, in cylindrical coordinates (r, ϑ, z) , the functions θ , f_s and f_g are independent of the angular coordinate ϑ , using the circular projection $(r, \vartheta, z) \mapsto (r, z)$ to reduce the model of Sec. 2 as well as the finite volume scheme to two space dimensions. As the arguments of [9, Sec. 3.6] are still valid in the present fully implicit case, analogous reasoning to the contents of the following Sec. 4 can be applied to the fully implicit axisymmetric finite volume scheme to prove a maximum principle as well as existence and uniqueness for the discrete solution, analogous to Th. 4.2, Cor. 1, and Cor. 2 below.

4. Existence and Uniqueness of a Discrete Solution to the Finite Volume Scheme, Maximum Principle. As the proof of existence and uniqueness of a discrete solution to the finite volume scheme in [9], the proof of existence and uniqueness of a discrete solution to the fully implicit finite volume scheme (35) in Th. 4.2 and Cor. 2 below is based on the root problem with maximum principle [9, Th. 4.1]. For the convenience of the reader, [9, Th. 4.1] is now reproduced as Th. 4.1:

Theorem 4.1. *Let $\tau \subseteq \mathbb{R}$ be a (closed, open, half-open, bounded or unbounded) interval. Given a finite, nonempty index set I , consider a continuous operator*

$$\mathcal{H} : \tau^I \longrightarrow \mathbb{R}^I, \quad \mathcal{H}(\mathbf{u}) = (\mathcal{H}_i(\mathbf{u}))_{i \in I}. \quad (38)$$

Assume there are continuous functions $b_i \in C(\tau, \mathbb{R})$, $\tilde{h}_i \in C(\tau, \mathbb{R})$, $\tilde{g}_i \in C(\tau^I, \mathbb{R})$, $i \in I$, such that the following conditions (i) – (iii) are satisfied.

(i) *There is $\tilde{\mathbf{u}} \in \tau^I$ such that, for each $i \in I$, $\mathbf{u} \in \tau^I$:*

$$\mathcal{H}_i(\mathbf{u}) = b_i(u_i) + \tilde{h}_i(u_i) - b_i(\tilde{u}_i) - \tilde{g}_i(\mathbf{u}).$$

(ii) *There are $\tilde{m}, \tilde{M} \in \tau$, a family of nonpositive numbers $(\beta_i)_{i \in I} \in (\mathbb{R}_0^-)^I$, and a family of nonnegative numbers $(B_i)_{i \in I} \in (\mathbb{R}_0^+)^I$ such that, for each $i \in I$, $\mathbf{u} \in \tau^I$, $\theta \in \tau$:*

$$\max \{ \|\mathbf{u}\|_{\max}, \tilde{M} \} \leq \theta \quad \Rightarrow \quad \tilde{g}_i(\mathbf{u}) - \tilde{h}_i(\theta) \leq B_i, \quad (39a)$$

$$\theta \leq \min \{ \tilde{m}, \|\mathbf{u}\|_{\min} \} \quad \Rightarrow \quad \tilde{g}_i(\mathbf{u}) - \tilde{h}_i(\theta) \geq \beta_i, \quad (39b)$$

where $\|\mathbf{u}\|_{\max}$ and $\|\mathbf{u}\|_{\min}$ are according to (27).

(iii) *There is a family of positive numbers $(C_{b,i})_{i \in I} \in (\mathbb{R}^+)^I$ such that, for each $i \in I$ and $\theta_1, \theta_2 \in \tau$:*

$$\theta_2 \geq \theta_1 \quad \Rightarrow \quad b_i(\theta_2) \geq (\theta_2 - \theta_1) C_{b,i} + b_i(\theta_1).$$

Letting

$$\beta := \min \left\{ \frac{\beta_i}{C_{b,i}} : i \in I \right\}, \quad B := \max \left\{ \frac{B_i}{C_{b,i}} : i \in I \right\}, \quad (40)$$

$$m(\tilde{\mathbf{u}}) := \min \{ \tilde{m}, \|\tilde{\mathbf{u}}\|_{\min} + \beta \}, \quad M(\tilde{\mathbf{u}}) := \max \{ \tilde{M}, \|\tilde{\mathbf{u}}\|_{\max} + B \}, \quad (41)$$

one has the following maximum principle: If $\mathbf{u}_0 \in \tau^I$ satisfies $\mathcal{H}(\mathbf{u}_0) = \mathbf{0} := (0, \dots, 0)$, then $\mathbf{u}_0 \in [m(\tilde{\mathbf{u}}), M(\tilde{\mathbf{u}})]^I$.

If, in addition to (i) – (iii), the following conditions (iv) – (vi) are satisfied, then there is a unique $\mathbf{u}_0 \in [m(\tilde{\mathbf{u}}), M(\tilde{\mathbf{u}})]^I$ such that $\mathcal{H}(\mathbf{u}_0) = \mathbf{0}$.

(iv) *For each $i \in I$, there is $L_{g,i}(\tilde{\mathbf{u}}) \in \mathbb{R}_0^+$ such that \tilde{g}_i is $L_{g,i}(\tilde{\mathbf{u}})$ -Lipschitz with respect to the max-norm on $[m(\tilde{\mathbf{u}}), M(\tilde{\mathbf{u}})]^I$.*

(v) *For each $i \in I$, there is $C_{\tilde{h},i}(\tilde{\mathbf{u}}) \in \mathbb{R}_0^+$ such that, for each $\theta_1, \theta_2 \in [m(\tilde{\mathbf{u}}), M(\tilde{\mathbf{u}})]$:*

$$\theta_2 \geq \theta_1 \quad \Rightarrow \quad \tilde{h}_i(\theta_2) \geq (\theta_2 - \theta_1) C_{\tilde{h},i}(\tilde{\mathbf{u}}) + \tilde{h}_i(\theta_1).$$

(vi) *$L_{g,i}(\tilde{\mathbf{u}}) < C_{b,i} + C_{\tilde{h},i}(\tilde{\mathbf{u}})$ for each $i \in I$.*

Proof. See [9, Th. 4.1]. □

As in [9], the essential step in proving the discrete existence and uniqueness results is to first provide a discrete existence result with maximum principle, locally in time. This is accomplished by the following Th. 4.2 which corresponds to [9, Th. 4.2]. Given an arbitrary vector $\tilde{\mathbf{u}} \in (\mathbb{R}_0^+)^I$, Th. 4.2 establishes that each root of the finite volume scheme operator $\mathcal{H}_\nu(\tilde{\mathbf{u}}, \cdot)$ of (36) satisfies a maximum principle. Moreover, Th. 4.2 proves the existence of a unique root to $\mathcal{H}_\nu(\tilde{\mathbf{u}}, \cdot)$, provided that the ν -th time step k_ν is sufficiently small.

As in [9, Th. 4.2], the upper and lower bound for the solution, respectively, given by (43e) and (43f) below, are determined by the external temperature θ_{ext} , by the max and min of $\tilde{\mathbf{u}}$ as defined in (27), by the size of the time step, and by the values of the heat sources in the time interval $[t_{\nu-1}, t_\nu]$.

The condition on the time step size (46) arises from the radiation terms in (36), namely, (36c) – (36e). It depends on the constants $L_{\mathbf{V}, \Gamma_\Omega}$, $L_{\mathbf{V}, \partial\Omega \setminus \Gamma_\Omega}$, and $L_{\mathbf{V}, \Sigma}$ defined in (43b) – (43d) below, involving the ratios between the size of boundary

elements and adjacent volume elements. Thus, these constants are of order h^{-1} if h is a parameter for the fineness of a space discretization constructed by uniform refinement of some initial grid, such that (46) is of the form $k \sim h$ in the notation of the Introduction.

Letting $\tilde{\mathbf{u}} = \mathbf{u}_{\nu-1}$, as a direct consequence of Th. 4.2, for k_ν small enough, each nonnegative solution $(\mathbf{u}_0, \dots, \mathbf{u}_{\nu-1})$ to the finite volume scheme (35) with N replaced by $\nu - 1 < N$, can be uniquely extended to $t = t_\nu$ (see Cor. 1).

Finally, as in [9, Th. 4.3], an inductive argument extends the local result of Th. 4.2 to guarantee a unique solution to the entire finite volume scheme (35) (see Cor. 2).

In preparation for Th. 4.2, notions of the *variation* of a function are recalled as well as some elementary properties: For a function $f : [a, \infty[\rightarrow \mathbb{R}$, let

$$\begin{aligned} \text{var}^+ f &: [a, \infty[\rightarrow [0, \infty], \\ \text{var}^+ f(a) &:= 0, \\ \text{var}^+ f(\lambda) &:= \sup \left\{ \sum_{\nu=1}^N \max \{0, f(t_\nu) - f(t_{\nu-1})\} : \right. \\ &\quad \left. (t_\nu)_{\nu \in \{0, \dots, N\}} \text{ is a discretization of } [a, \lambda] \right\} \text{ for } \lambda > a, \end{aligned} \quad (42)$$

denote the *positive variation* of f , and define its *negative variation* $\text{var}^- f$ by replacing “max” in (42) with “− min”. Then, $\text{var}^+ f$ and $\text{var}^- f$ are nonnegative and increasing. Moreover, if f is L -Lipschitz on $[a, r]$, then $\text{var}^+ f$ and $\text{var}^- f$ are L -Lipschitz on $[a, r]$, and, for each $\lambda \in [a, r]$, $f(\lambda) = f(a) + \text{var}^+ f(\lambda) - \text{var}^- f(\lambda)$.

Theorem 4.2. Assume (A-1) – (A-8), (DA-1) – (DA-6), (AA-1) and (AA-2). Moreover, assume $\nu \in \{1, \dots, N\}$ and $\tilde{\mathbf{u}} = (\tilde{u}_i)_{i \in I} \in (\mathbb{R}_0^+)^I$. Let

$$B_{f,\nu} := \max \left\{ \sum_{m \in \{s,g\}} f_{m,\nu,i} \frac{\lambda_3(\omega_{m,i})}{\lambda_3(\omega_i)} : i \in I \right\}, \quad (43a)$$

$$L_{\mathbf{V}, \Gamma_\Omega} := \sigma \max \left\{ \frac{\lambda_2(\partial\omega_{s,i} \cap \Gamma_\Omega)}{\lambda_3(\omega_i)} : i \in I \right\}, \quad (43b)$$

$$L_{\mathbf{V}, \partial\Omega \setminus \Gamma_\Omega} := \sigma \max \left\{ \frac{\lambda_2(\partial\omega_{s,i} \cap (\partial\Omega \setminus \Gamma_\Omega))}{\lambda_3(\omega_i)} : i \in I \right\}, \quad (43c)$$

$$L_{\mathbf{V}, \Sigma} := \sigma \max \left\{ \frac{\lambda_2(\omega_i \cap \Sigma)}{\lambda_3(\omega_i)} : i \in I \right\}, \quad (43d)$$

$$m(\tilde{\mathbf{u}}) := \min \{ \theta_{\text{ext}}, \|\tilde{\mathbf{u}}\|_{\min} \}, \quad (43e)$$

$$M_\nu(\tilde{\mathbf{u}}) := \max \left\{ \theta_{\text{ext}}, \|\tilde{\mathbf{u}}\|_{\max} + \frac{k_\nu}{C_\varepsilon} B_{f,\nu} \right\}, \quad (43f)$$

with $\|\tilde{\mathbf{u}}\|_{\min}$, $\|\tilde{\mathbf{u}}\|_{\max}$ according to (27), and C_ε according to (A-3).

Then, one has the maximum principle that each solution $\mathbf{u}_\nu = (u_{\nu,i})_{i \in I} \in (\mathbb{R}_0^+)^I$ to

$$\mathcal{H}_{\nu,i}(\tilde{\mathbf{u}}, \mathbf{u}_\nu) = 0 \quad (i \in I) \quad (44)$$

must lie in $[m(\tilde{\mathbf{u}}), M_\nu(\tilde{\mathbf{u}})]^I$. Furthermore, if

$$l_g^1 : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+, \quad l_g^1(r) := 4 \operatorname{var}^- \epsilon(r) r^3 + L_{\epsilon,r} r^4, \quad (45a)$$

$$l_g^2 : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+, \quad l_g^2(r) := 4 \epsilon_{\min,r}^{-1} r^3 + \max\{r^4, \theta_{\text{ext}}^4\} L_{\epsilon,r} (2 \epsilon_{\min,r}^{-1} + 1), \quad (45b)$$

$$l_g^3 : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+, \quad l_g^3(r) := L_{\epsilon,r} \theta_{\text{ext}}^4, \quad (45c)$$

where $L_{\epsilon,r}$ and $\epsilon_{\min,r}$ are according to (A-6) and Rem. 2, respectively, and, if k_ν is such that

$$\begin{aligned} k_\nu \Big((L_{\mathbf{V},\Gamma_\Omega} + L_{\mathbf{V},\partial\Omega \setminus \Gamma_\Omega} + L_{\mathbf{V},\Sigma}) l_g^1(M_\nu(\tilde{\mathbf{u}})) + (L_{\mathbf{V},\Gamma_\Omega} + L_{\mathbf{V},\Sigma}) l_g^2(M_\nu(\tilde{\mathbf{u}})) \\ + L_{\mathbf{V},\partial\Omega \setminus \Gamma_\Omega} l_g^3(M_\nu(\tilde{\mathbf{u}})) \Big) < C_\epsilon, \end{aligned} \quad (46)$$

then there is a unique $\mathbf{u}_\nu \in [m(\tilde{\mathbf{u}}), M_\nu(\tilde{\mathbf{u}})]^I$ satisfying (44).

Proof. First note that, by choosing k_ν sufficiently small, one can ensure that (46) is satisfied: Since all three functions l_g^1, l_g^2, l_g^3 are increasing, it follows from (43f) that, by decreasing k_ν , one decreases both factors on the left-hand side of (46).

Now, the goal is to apply Th. 4.1 with $\tau = \mathbb{R}_0^+$ and $\mathcal{H}_\nu(\tilde{\mathbf{u}}, \cdot)$ playing the role of \mathcal{H} . To that end, in the following, one defines continuous functions $b_{\nu,i}, \tilde{h}_i, \tilde{g}_{\nu,i}$, as well as numbers $\tilde{m}, \tilde{M} \in \mathbb{R}_0^+, \beta_i \in \mathbb{R}_0^-, B_{\nu,i} \in \mathbb{R}_0^+, C_{b,\nu,i} \in \mathbb{R}^+, L_{g,\nu,i}(\tilde{\mathbf{u}}) \in \mathbb{R}^+$, and $C_{\tilde{h},\nu,i}(\tilde{\mathbf{u}}) \in \mathbb{R}^+$ that satisfy the hypotheses of Th. 4.1 (where the quantities with index ν correspond to the matching quantities without index ν in Th. 4.1). Condition (46) will *only* be needed to prove hypothesis (vi) of Th. 4.1.

For each $i \in I$, let

$$b_{\nu,i} : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+, \quad b_{\nu,i}(\theta) := k_\nu^{-1} \sum_{m \in \{\text{s,g}\}} \varepsilon_m(\theta) \lambda_3(\omega_{m,i}), \quad (47a)$$

$$L_{\kappa,i} := \sum_{m \in \{\text{s,g}\}} \kappa_m \sum_{j \in \text{nb}_m(i)} \frac{\lambda_2(\partial\omega_{m,i} \cap \partial\omega_{m,j})}{\|x_i - x_j\|_2} \geq 0, \quad (47b)$$

$$C_{\mathbf{V},i} := \sigma \lambda_2(\partial\omega_{\text{s},i} \cap \Gamma_\Omega) + \sigma \lambda_2(\partial\omega_{\text{s},i} \cap (\partial\Omega \setminus \Gamma_\Omega)) + \sigma \lambda_2(\omega_i \cap \Sigma) \geq 0, \quad (47c)$$

$$\begin{aligned} \tilde{h}_i : \mathbb{R}_0^+ \longrightarrow \mathbb{R}, \\ \tilde{h}_i(\theta) := L_{\kappa,i} \theta + C_{\mathbf{V},i} (\epsilon(0) + \operatorname{var}^+ \epsilon(\theta)) \theta^4 \\ + \sigma (\operatorname{var}^- \epsilon(\theta) - \epsilon(0)) \theta_{\text{ext}}^4 \lambda_2(\partial\omega_{\text{s},i} \cap (\partial\Omega \setminus \Gamma_\Omega)), \end{aligned} \quad (47d)$$

$$\begin{aligned} \tilde{g}_{\nu,i} : (\mathbb{R}_0^+)^I \longrightarrow \mathbb{R}_0^+, \\ \tilde{g}_{\nu,i}(\mathbf{u}) := \sum_{m \in \{\text{s,g}\}} \kappa_m \sum_{j \in \text{nb}_m(i)} \frac{u_j}{\|x_i - x_j\|_2} \lambda_2(\partial\omega_{m,i} \cap \partial\omega_{m,j}) \\ + C_{\mathbf{V},i} \operatorname{var}^- \epsilon(u_i) u_i^4 \\ + \sum_{\alpha \in J_{\Omega,i}} V_{\Gamma,\alpha}(\mathbf{u}|_{I_\Omega}) + \sigma \operatorname{var}^+ \epsilon(u_i) \theta_{\text{ext}}^4 \lambda_2(\partial\omega_{\text{s},i} \cap (\partial\Omega \setminus \Gamma_\Omega)) \\ + \sum_{\alpha \in J_{\Sigma,i}} V_{\Sigma,\alpha}(\mathbf{u}|_{I_\Sigma}) + \sum_{m \in \{\text{s,g}\}} f_{m,\nu,i} \lambda_3(\omega_{m,i}), \end{aligned} \quad (47e)$$

$$\tilde{m} := \tilde{M} := \theta_{\text{ext}}, \quad \beta_i := 0, \quad B_{\nu,i} := \sum_{m \in \{s,g\}} f_{m,\nu,i} \lambda_3(\omega_{m,i}), \quad (47f)$$

$$C_{b,\nu,i} := k_\nu^{-1} C_\varepsilon \lambda_3(\omega_i) > 0, \quad (47g)$$

$$\begin{aligned} L_{g,\nu,i}(\tilde{\mathbf{u}}) &:= L_{\kappa,i} + C_{\mathbf{V},i} l_g^1(M_\nu(\tilde{\mathbf{u}})) + \sigma (\lambda_2(\partial\omega_{s,i} \cap \Gamma_\Omega) + \lambda_2(\omega_i \cap \Sigma)) l_g^2(M_\nu(\tilde{\mathbf{u}})) \\ &\quad + \sigma \lambda_2(\partial\omega_{s,i} \cap (\partial\Omega \setminus \Gamma_\Omega)) l_g^3(M_\nu(\tilde{\mathbf{u}})) \geq 0, \end{aligned} \quad (47h)$$

$$C_{h,\nu,i}(\tilde{\mathbf{u}}) := C_{b,\nu,i} + L_{\kappa,i} + 4m(\tilde{\mathbf{u}})^3 \epsilon(0) C_{\mathbf{V},i} > 0. \quad (47i)$$

Note that the numbers $m(\tilde{\mathbf{u}})$ and $M_\nu(\tilde{\mathbf{u}})$ defined in (43e) and (43f), respectively, correspond to the numbers $m(\tilde{\mathbf{u}})$ and $M(\tilde{\mathbf{u}})$ as defined in (41) in Th. 4.1.

It remains to verify the hypotheses (i) – (vi) of Th. 4.1.

Th. 4.1(i): Showing $\mathcal{H}_{\nu,i}(\tilde{\mathbf{u}}, \mathbf{u}) = b_{\nu,i}(u_i) + \tilde{h}_i(u_i) - b_{\nu,i}(\tilde{u}_i) - \tilde{g}_{\nu,i}(\mathbf{u})$ is straightforward from the respective definitions in (47).

Th. 4.1(ii): One has to show that, for each $i \in I$, $\mathbf{u} \in (\mathbb{R}_0^+)^I$, $\theta \in \mathbb{R}_0^+$:

$$\max \{ \|\mathbf{u}\|_{\max}, \theta_{\text{ext}} \} \leq \theta \quad \Rightarrow \quad \tilde{g}_{\nu,i}(\mathbf{u}) - \tilde{h}_i(\theta) \leq B_{\nu,i}, \quad (48a)$$

$$\theta \leq \min \{ \theta_{\text{ext}}, \|\mathbf{u}\|_{\min} \} \quad \Rightarrow \quad \tilde{g}_{\nu,i}(\mathbf{u}) - \tilde{h}_i(\theta) \geq 0. \quad (48b)$$

Considering Lem. 3.2(b) and Rem. 4, one sees that

$$\begin{aligned} \sum_{\alpha \in J_{\Omega,i}} V_{\Gamma,\alpha}(\mathbf{u}|_{I_\Omega}) &\leq \sigma \epsilon(u_i) \max \{ \|\mathbf{u}\|_{\max}^4, \theta_{\text{ext}}^4 \} \lambda_2(\partial\omega_{s,i} \cap \Gamma_\Omega), \\ \sum_{\alpha \in J_{\Sigma,i}} V_{\Sigma,\alpha}(\mathbf{u}|_{I_\Sigma}) &\leq \sigma \epsilon(u_i) \|\mathbf{u}\|_{\max}^4 \lambda_2(\omega_i \cap \Sigma). \end{aligned}$$

If $\theta \geq \theta_{\text{ext}}$ and $\theta \geq \|\mathbf{u}\|_{\max}$, then, noting that $\epsilon(\theta) + \text{var}^- \epsilon(\theta) = \epsilon(0) + \text{var}^+ \epsilon(\theta)$, and, by recalling (43a) and (47b) – (47f), one obtains

$$\begin{aligned} \tilde{g}_{\nu,i}(\mathbf{u}) &\leq \sum_{m \in \{s,g\}} \kappa_m \sum_{j \in \text{nb}_m(i)} \frac{\theta}{\|x_i - x_j\|_2} \lambda_2(\partial\omega_{m,i} \cap \partial\omega_{m,j}) \\ &\quad + C_{\mathbf{V},i} \text{var}^- \epsilon(\theta) \theta^4 \\ &\quad + \sigma \epsilon(\theta) \theta^4 \lambda_2(\partial\omega_{s,i} \cap \Gamma_\Omega) \\ &\quad + \sigma \epsilon(\theta) (\theta^4 - \theta_{\text{ext}}^4) \lambda_2(\partial\omega_{s,i} \cap (\partial\Omega \setminus \Gamma_\Omega)) \\ &\quad + \sigma \text{var}^+ \epsilon(\theta) \theta_{\text{ext}}^4 \lambda_2(\partial\omega_{s,i} \cap (\partial\Omega \setminus \Gamma_\Omega)) \\ &\quad + \sigma \epsilon(\theta) \theta^4 \lambda_2(\omega_i \cap \Sigma) + \sum_{m \in \{s,g\}} f_{m,\nu,i} \lambda_3(\omega_{m,i}) \\ &= \theta L_{\kappa,i} + C_{\mathbf{V},i} (\epsilon(0) + \text{var}^+ \epsilon(\theta)) \theta^4 \\ &\quad + \sigma (\text{var}^- \epsilon(\theta) - \epsilon(0)) \theta_{\text{ext}}^4 \lambda_2(\partial\omega_{s,i} \cap (\partial\Omega \setminus \Gamma_\Omega)) + \sum_{m \in \{s,g\}} f_{m,\nu,i} \lambda_3(\omega_{m,i}) \\ &= \tilde{h}_i(\theta) + B_{\nu,i}, \end{aligned}$$

proving (48a). On the other hand, if $\theta \leq \theta_{\text{ext}}$ and $\theta \leq \|\mathbf{u}\|_{\min}$, then, as $f_{m,\nu,i} \geq 0$ by (AA-1), an analogous computation shows $\tilde{g}_{\nu,i}(\mathbf{u}) \geq \tilde{h}_i(\theta)$, proving (48b).

Th. 4.1(iii): That, for $\theta_2 \geq \theta_1 \geq 0$, one has $b_{\nu,i}(\theta_2) \geq (\theta_2 - \theta_1) C_{b,\nu,i} + b_{\nu,i}(\theta_1)$ is immediate from combining (A-3), (47a), and (47g).

Th. 4.1(iv): For each $i \in I$, one has to show that $\tilde{g}_{\nu,i}$ is $L_{g,\nu,i}(\tilde{\mathbf{u}})$ -Lipschitz with respect to the max-norm on $[m(\tilde{\mathbf{u}}), M_\nu(\tilde{\mathbf{u}})]^I$. The function

$$\mathbf{u} \mapsto \sum_{m \in \{s,g\}} \kappa_m \sum_{j \in \text{nb}_m(i)} \frac{u_j}{\|x_i - x_j\|_2} \lambda_2(\partial\omega_{m,i} \cap \partial\omega_{m,j}) \quad (\mathbf{u} \in (\mathbb{R}_0^+)^I)$$

is $L_{\kappa,i}$ -Lipschitz, $L_{\kappa,i}$ according to (47b), and, using (45a), the map

$$\mathbf{u} \mapsto C_{\mathbf{V},i} \text{var}^- \epsilon(u_i) u_i^4 \quad (\mathbf{u} \in [0, M_\nu(\tilde{\mathbf{u}})]^I)$$

is $(C_{\mathbf{V},i} l_g^1(M_\nu(\tilde{\mathbf{u}})))$ -Lipschitz by (A-6). Furthermore, Lem. 3.2(c) and Rem. 4 show that the function $\sum_{\alpha \in J_{\Omega,i}} V_{\Gamma,\alpha}$ is $(\sigma \lambda_2(\partial\omega_{s,i} \cap \Gamma_\Omega) l_g^2(M_\nu(\tilde{\mathbf{u}})))$ -Lipschitz on $[0, M_\nu(\tilde{\mathbf{u}})]^{I_\Omega}$ and that the function $\sum_{\alpha \in J_{\Sigma,i}} V_{\Sigma,\alpha}$ is $(\sigma \lambda_2(\omega_i \cap \Sigma) l_g^2(M_\nu(\tilde{\mathbf{u}})))$ -Lipschitz on $[0, M_\nu(\tilde{\mathbf{u}})]^{I_\Sigma}$. Finally, combining (45c) with (A-6) yields that $\text{var}^+ \epsilon \theta_{\text{ext}}^4$ is $l_g^3(M_\nu(\tilde{\mathbf{u}}))$ -Lipschitz, such that, by (47e) and (47h), $\tilde{g}_{\nu,i}$ is $L_{g,\nu,i}(\tilde{\mathbf{u}})$ -Lipschitz on $[m(\tilde{\mathbf{u}}), M_\nu(\tilde{\mathbf{u}})]^I$ as needed.

Th. 4.1(v): Let $i \in I$ and $M_\nu(\tilde{\mathbf{u}}) \geq \theta_2 \geq \theta_1 \geq m(\tilde{\mathbf{u}})$. One needs to show that $\tilde{h}_i(\theta_2) \geq (\theta_2 - \theta_1) (L_{\kappa,i} + 4m(\tilde{\mathbf{u}})^3 \epsilon(0) C_{\mathbf{V},i}) + \tilde{h}_i(\theta_1)$. Since $\theta \mapsto \theta^4$ is a convex function on \mathbb{R}_0^+ , one has $\theta_2^4 \geq 4m(\tilde{\mathbf{u}})^3 (\theta_2 - \theta_1) + \theta_1^4$. As $\text{var}^+ \epsilon$ and $\text{var}^- \epsilon$ are increasing, (47d) yields

$$\begin{aligned} \tilde{h}_i(\theta_2) &\geq (\theta_2 - \theta_1) \left(L_{\kappa,i} + 4m(\tilde{\mathbf{u}})^3 \sigma \epsilon(0) \left(\lambda_2(\partial\omega_{s,i} \cap \Gamma_\Omega) \right. \right. \\ &\quad \left. \left. + \lambda_2(\partial\omega_{s,i} \cap (\partial\Omega \setminus \Gamma_\Omega)) + \lambda_2(\omega_i \cap \Sigma) \right) \right) \\ &\quad + \theta_1 L_{\kappa,i} + \sigma (\epsilon(0) + \text{var}^+ \epsilon(\theta_1)) \theta_1^4 \lambda_2(\partial\omega_{s,i} \cap \Gamma_\Omega) \\ &\quad + \sigma (\epsilon(0) + \text{var}^+ \epsilon(\theta_1)) \theta_1^4 \lambda_2(\partial\omega_{s,i} \cap (\partial\Omega \setminus \Gamma_\Omega)) \\ &\quad + \sigma (\epsilon(0) + \text{var}^+ \epsilon(\theta_1)) \theta_1^4 \lambda_2(\omega_i \cap \Sigma) \\ &\quad + \sigma (\text{var}^- \epsilon(\theta_1) - \epsilon(0)) \theta_{\text{ext}}^4 \lambda_2(\partial\omega_{s,i} \cap (\partial\Omega \setminus \Gamma_\Omega)) \\ &= (\theta_2 - \theta_1) (L_{\kappa,i} + 4m(\tilde{\mathbf{u}})^3 \epsilon(0) C_{\mathbf{V},i}) + \tilde{h}_i(\theta_1), \end{aligned}$$

thereby establishing the case.

Th. 4.1(vi): For each $i \in I$, one has to show that $L_{g,\nu,i}(\tilde{\mathbf{u}}) < C_{h,\nu,i}(\tilde{\mathbf{u}})$, where $L_{g,\nu,i}(\tilde{\mathbf{u}})$ and $C_{h,\nu,i}(\tilde{\mathbf{u}})$ are according to (47h) and (47i), respectively. The desired inequality follows from (46) by the following calculation:

$$\begin{aligned} L_{g,\nu,i}(\tilde{\mathbf{u}}) &= L_{\kappa,i} + C_{\mathbf{V},i} l_g^1(M_\nu(\tilde{\mathbf{u}})) \\ &\quad + \sigma (\lambda_2(\partial\omega_{s,i} \cap \Gamma_\Omega) + \lambda_2(\omega_i \cap \Sigma)) l_g^2(M_\nu(\tilde{\mathbf{u}})) \\ &\quad + \sigma \lambda_2(\partial\omega_{s,i} \cap (\partial\Omega \setminus \Gamma_\Omega)) l_g^3(M_\nu(\tilde{\mathbf{u}})) \\ &\stackrel{(43b)-(43d)}{\leq} L_{\kappa,i} + \lambda_3(\omega_i) (L_{\mathbf{V},\Gamma_\Omega} + L_{\mathbf{V},\partial\Omega \setminus \Gamma_\Omega} + L_{\mathbf{V},\Sigma}) l_g^1(M_\nu(\tilde{\mathbf{u}})) \\ &\quad + \lambda_3(\omega_i) (L_{\mathbf{V},\Gamma_\Omega} + L_{\mathbf{V},\Sigma}) l_g^2(M_\nu(\tilde{\mathbf{u}})) \\ &\quad + \lambda_3(\omega_i) L_{\mathbf{V},\partial\Omega \setminus \Gamma_\Omega} l_g^3(M_\nu(\tilde{\mathbf{u}})) + 4m(\tilde{\mathbf{u}})^3 \epsilon(0) C_{\mathbf{V},i} \\ &\stackrel{(46)}{<} k_\nu^{-1} C_\epsilon \lambda_3(\omega_i) + L_{\kappa,i} + 4m(\tilde{\mathbf{u}})^3 \epsilon(0) C_{\mathbf{V},i} \\ &= C_{b,\nu,i} + L_{\kappa,i} + 4m(\tilde{\mathbf{u}})^3 \epsilon(0) C_{\mathbf{V},i} \\ &= C_{h,\nu,i}(\tilde{\mathbf{u}}). \end{aligned}$$

Hence, all hypotheses of Th. 4.1 are verified, and the conclusion of Th. 4.1 provides a unique vector $\mathbf{u}_\nu \in [m(\tilde{\mathbf{u}}), M_\nu(\tilde{\mathbf{u}})]^I$ such that $\mathcal{H}_{\nu,i}(\tilde{\mathbf{u}}, \mathbf{u}_\nu) = 0$ for each

$i \in I$. Since Th. 4.1 also yields that \mathbf{u}_ν is the only element of $(\mathbb{R}_0^+)^I$ satisfying $\mathcal{H}_{\nu,i}(\tilde{\mathbf{u}}, \mathbf{u}_\nu) = 0$ for each $i \in I$, the proof of Th. 4.2 is complete. \square

Corollary 1. Assume (A-1) – (A-8), (DA-1) – (DA-6), (AA-1), (AA-2), and let $(\mathbf{u}_0, \dots, \mathbf{u}_{n-1})$, $n \leq N$, $\mathbf{u}_\nu = (u_{\nu,i})_{i \in I}$, be a nonnegative solution to (35) (where N is replaced by $n - 1$). Then each solution $\mathbf{u}_n \in (\mathbb{R}_0^+)^I$ to $\mathcal{H}_{n,i}(\mathbf{u}_{n-1}, \mathbf{u}_n) = 0$ (for each $i \in I$), where $\mathcal{H}_{n,i}$ is defined by (36), must lie in $[m(\mathbf{u}_{n-1}), M_n(\mathbf{u}_{n-1})]^I$, with $m(\mathbf{u}_{n-1})$ and $M_n(\mathbf{u}_{n-1})$ defined according to (43e) and (43f), respectively. Furthermore, if k_n satisfies condition (46), then there is a unique $\mathbf{u}_n \in (\mathbb{R}_0^+)^I$ that satisfies $\mathcal{H}_{n,i}(\mathbf{u}_{n-1}, \mathbf{u}_n) = 0$ for each $i \in I$.

Corollary 2. Assume (A-1) – (A-8), (DA-1) – (DA-6), (AA-1) and (AA-2). Let

$$m := \min \{ \theta_{\text{ext}}, \text{ess inf}(\theta_{\text{init}}) \}, \quad (49)$$

$$M_\nu := \max \{ \theta_{\text{ext}}, \|\theta_{\text{init}}\|_{L^\infty(\Omega)} \} + \frac{t_\nu}{C_\varepsilon} \sum_{m \in \{s, g\}} \|f_m\|_{L^\infty([0, t_\nu] \times \Omega_m)} \quad (50)$$

for each $\nu \in \{0, \dots, N\}$.

If $(\mathbf{u}_0, \dots, \mathbf{u}_N) = (u_{\nu,i})_{(\nu,i) \in \{0, \dots, N\} \times I} \in (\mathbb{R}_0^+)^{I \times \{0, \dots, N\}}$ is a solution to the finite volume scheme (35), then $\mathbf{u}_\nu \in [m, M_\nu]^I$ for each $\nu \in \{0, \dots, N\}$. Furthermore, if

$$\begin{aligned} k_\nu & \left((L_{\mathbf{V}, \Gamma_\Omega} + L_{\mathbf{V}, \partial\Omega \setminus \Gamma_\Omega} + L_{\mathbf{V}, \Sigma}) l_g^1(M_\nu) + (L_{\mathbf{V}, \Gamma_\Omega} + L_{\mathbf{V}, \Sigma}) l_g^2(M_\nu) \right. \\ & \left. + L_{\mathbf{V}, \partial\Omega \setminus \Gamma_\Omega} l_g^3(M_\nu) \right) < C_\varepsilon \quad (\nu \in \{1, \dots, N\}), \end{aligned} \quad (51)$$

where $L_{\mathbf{V}, \Gamma_\Omega}$, $L_{\mathbf{V}, \partial\Omega \setminus \Gamma_\Omega}$, $L_{\mathbf{V}, \Sigma}$, l_g^1 , l_g^2 , and l_g^3 are defined according to (43) and (45), respectively, then the finite volume scheme (35) has a unique solution $(\mathbf{u}_0, \dots, \mathbf{u}_N) \in (\mathbb{R}_0^+)^{I \times \{0, \dots, N\}}$. It is pointed out that a sufficient condition for (51) to be satisfied is

$$\begin{aligned} \max \{ k_\nu : \nu \in \{1, \dots, N\} \} & \left((L_{\mathbf{V}, \Gamma_\Omega} + L_{\mathbf{V}, \partial\Omega \setminus \Gamma_\Omega} + L_{\mathbf{V}, \Sigma}) l_g^1(M_N) \right. \\ & \left. + (L_{\mathbf{V}, \Gamma_\Omega} + L_{\mathbf{V}, \Sigma}) l_g^2(M_N) \right. \\ & \left. + L_{\mathbf{V}, \partial\Omega \setminus \Gamma_\Omega} l_g^3(M_N) \right) < C_\varepsilon. \end{aligned} \quad (52)$$

Proof. The proof can be carried out by induction on $n \in \{0, \dots, N\}$ analogous to the proof of [9, Th. 4.3]. \square

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